NONDEGENERATE SURFACES OF FINITE TOPOLOGICAL TYPE[1]

BY

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1. **Introduction.** In the theory of surface area and its applications to the calculus of variations Morrey's theorem plays an important role. The purpose of this paper is to generalize this result to surfaces of the topological type of any compact 2-manifold, orientable or not, with or without boundary:

THEOREM. *Every nondegenerate Fréchet surface of any finite topological type* τ *with finite Lebesgue area has a quasi-conformal representation* $T(w)$. *We may take as the domain of* $T(w)$ *a certain normalized parallel slit domain.*

For 2-cells and 2-spheres this is Morrey's theorem [7], also [1, p. 484]. We shall suppose here that τ is neither of these types. The proof strongly uses the result that a polyhedron of higher topological type has a quasi-conformal representation on a normalized parallel slit domain [3, Chapter II][2]. This replaces a classical result for the 2-cell case due to Schwarz. Slit domains were used by Courant [3] and Shiffman [11] in their work on the Plateau problem. In fact, several lemmas in the present paper have counterparts in [11].

For surfaces defined as mappings from any admissible plane set [1, p. 27] Morrey's theorem was used by Cesari in the proof that any such surface with finite Lebesgue area has a generalized conformal representation [1, p. 487] with absolutely continuous plane projections [1, p. 544, the third theorem]. For surfaces of the type of the 2-cell or 2-sphere Morrey's theorem has been used by Radó [9, especially p. 501] together with cyclic decompositions to prove that the Lebesgue area $L(S)$ equals the lower area defined in [9] provided $L(S)$ is finite. Cesari [1] has since shown by other methods that this equality holds even if $L(S) = \infty$, for surfaces defined as mappings from any admissible plane set. Neugebauer and the author plan to give a new proof of Cesari's result (under the assumption $L(S) < \infty$) based on the above extension of Morrey's theorem and Cesari's fine cyclic decompositions of surfaces [2]. The fine cyclic elements may be slightly degenerate and need not have any quasi-conformal representation (see §8). However, any fine cyclic element becomes nondegenerate upon cutting it at a finite number of points at most.

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Methods similar to the ones in this paper are used by the author else-where (to appear) to obtain representations of generalized surfaces of finite topological types (see Young [13]), and thereby in connection with para-metric problems of the calculus of variations.

<center>PURELY TOPOLOGICAL PART</center>

2. **Definitions and preliminaries.** The letter A will stand for a 2-manifold [14, p. 25], w a point of A, and $T(w)$ a continuous mapping from A to euclid-ean N-space E_N ($N \geqq 2$). A may be closed, or may have disjoint boundary curves $C_1, \cdots, C_r$. Let q denote the characteristic of A. If A is orientable, then A is a topological sphere with ν holes and h handles, where

$$h = (q - \nu + 2)/2.$$

In the nonorientable case, A is a sphere with ν holes and $k = q - \nu + 2$ cross-caps. The quantity $\tau = (\nu, q, \epsilon)$, where ϵ denotes orientability or not, will be called *topological type* of A. The boundary $C_1 \cup \cdots \cup C_\nu$ of A is denoted by A^*. If B is a subset of A, B^0 and B^- denote, respectively, its interior and closure relative to A. We use the term *relative boundary* of B for $B^- \cap (A - B)^-$.

By *cross-cut* we mean an arc in A with endpoints on A^* not meeting A^* between endpoints. Let $\Omega(A)$ denote the set of all closed curves in A not null homotopic, together with all cross-cuts γ such that there is no arc λ of A^* with the same endpoints as γ such that $\gamma \cup \lambda$ is null homotopic. Clearly every cross-cut with endpoints on different boundary curves C_i, C_j belongs to $\Omega(A)$. The following known result will often be used:

LEMMA 1. *A simple closed curve γ in A is null homotopic if and only if γ bounds a 2-cell $Z \subset A$.*

In particular, every simple closed curve or cross-cut γ which does not separate A belongs to $\Omega(A)$, unless A is a 2-cell and $\gamma = A^*$.

Let S denote a Fréchet surface in E_N, of type τ. We consider only non-oriented surfaces. However, if τ is an orientable type one of the two possible orientations may be assigned and the discussion still applies with only minor modifications. $\|S, S'\|$ denotes Fréchet distance, and is of course only defined if S and S' are of the same topological type.

For brevity we say (T, A) represents S, or simply T represents S, to mean T is a representation of S with domain A. Let $\Gamma(T, A)$ denote the set of all maximal continua of constancy of T.

(2.1) DEFINITION. A surface S is *nondegenerate* if, for every representation (T, A) of S:

(a) Given $g \in \Gamma(T, A)$ there exists an open set G containing g such that no $\gamma \in \Omega(A)$ is entirely contained in G;

(b) No $g \in \Gamma(T, A)$ separates A (i.e., $A - g$ is connected for every g); and

(c) T is not constant on A^*; or, if A is closed, T is not constant on A.

For 2-cells and 2-spheres this definition agrees with that of open and closed nondegenerate surface, respectively, given by Cesari [1, p. 472]. If S is neither of these types (a) implies (c). For each $\epsilon > 0$ let $\Phi(\epsilon)$ denote the set of all continua K separating A such that diam $T(G) > \epsilon$ for at least two components G of $A - K$. Let

$$\phi(S) = \inf_{\gamma \in \Omega(A)} \text{diam } T(\gamma),$$

(2.2)

$$\psi(S, \epsilon) = \inf_{K \in \Phi(\epsilon)} \text{diam } T(K),$$

where (T, A) is any representation of S.

If $\Omega(A)$ is empty we set $\phi(S) = +\infty$, and similarly if $\Phi(\epsilon)$ is empty then $\psi(S, \epsilon) = +\infty$. The indicated Fréchet invariance is easily verified. $\phi(S)$ is continuous. This will be needed later, and also the useful inequality:

(2.3) $\psi(S, \epsilon - \sigma) \leqq \psi(S', \epsilon) + \sigma$, if $\|S, S'\| < \sigma/2$.

LEMMA 2. *S is nondegenerate if and only if* $\phi(S) > 0$, $\psi(S, \epsilon) > 0$ *for every* $\epsilon > 0$, *and in case S is of the type of the 2-cell or 2-sphere* (2.1c) *holds.*

Proof. *Necessity.* The proof that $\psi(S, \epsilon) > 0$ is essentially the same as for the special case [1, 33.4(i)]. If we assume $\phi(S) = 0$, then by Zoretti's theorem [1, (10.2)] there exist $\gamma_n \in \Omega(A)$ and a continuum K on which T is constant such that infinitely many γ_n belong to any open set containing K. This is contrary to (2.1 a). *Sufficiency.* Given g take in (2.1 a) G as the set of w for which $|T(w) - T(g)| < \phi(S)$. Obviously (2.1 b) holds, and except when A is a 2-cell or 2-sphere (a) implies (c).

3. **Continuation.** The following theorem, taken together with an important approximation theorem of J. W. T. Youngs [15] proves that S is nondegenerate if and only if S has a light representation. Therefore, our definition is equivalent to that used by Morrey [7] and Radó [9]. This theorem will not be used in proving the main result concerning quasi-conformal representations.

THEOREM. *A surface S is nondegenerate of type* τ *if and only if its middle space is a 2-manifold of type* τ.

Proof. *Sufficiency.* This is immediate from the fact that S has a light representation if A and M are homeomorphic, which follows from Youngs' approximation theorem just cited.

Necessity. Suppose first that A is closed. By results of Roberts and Steenrod [10], see also Youngs [14, (6.4) and (6.5)], M is homeomorphic with A if (and only if) for every $g \in \Gamma$ and open set G containing g there exists a 2-cell Z such that $g \subset Z^0$ and $Z \subset G$. Consider $g \in \Gamma$ and G an open set containing g. Using (a) let G' be an open subset of G such that $g \subset G'$ and every closed curve in G' is null homotopic in A. By (b), $U = A - g$ is connected. By a lemma of Roberts and Steenrod [10; 14] there exist disjoint simple

closed curves $\lambda_1, \cdots, \lambda_m$ contained in $G'-g$ such that $\lambda_1 \cup \cdots \cup \lambda_m$ bounds a 2-manifold B contained in U and for each $i=1, \cdots, m$ there is an open cylinder B_i, bounded by λ_i and a portion of g, such that $B_i^- \subset G'$. The set $Z = A - B = B_1 \cup \cdots \cup B_m \cup g$ is connected, since each point of Z can be joined in Z to the continuum g. Then $m=1$; otherwise, since both Z and B are connected, no single λ_i separates A. Then $\lambda_i \subset G'$ is not null homotopic in A, contrary to choice of G'. Similarly, since $Z \subset G'$ every simple closed curve in Z separates A, therefore also separates the open set Z. Thus Z is an open 2-cell bounded by λ_1. Clearly, $g \subset Z$ and $Z^- \subset G$. Since g and G were chosen arbitrarily, M is homeomorphic with A [3].

Before considering surfaces with boundary we prove the following:

LEMMA 3. *Let (T, A) represent a nondegenerate surface S. Then for any $g \in \Gamma$, $g \cap A^*$ is either void, a point, or an arc.*

Proof. Since no g contains any component of A^*, it suffices to show that $g \cap A^*$ is connected or void for any g. Suppose there were two points w_1 and w_2 belonging to different components of $g \cap A^*$. By known theorems: given $\delta > 0$, w_1 and w_2 can be joined by a cross-cut γ such that diam $T(\gamma) < \delta$. If $\delta < \phi(S)$, then γ together with an arc λ of a single component C_i of A^* bound a 2-cell by Lemmas 1 and 2. Choose $\epsilon = \min[\text{diam } T(\lambda), \text{ diam } T(C_i - \lambda)]/2$, and $\delta < \psi(S, \epsilon)$. By Lemma 2 this is a contradiction.

Now let A have boundaries $C_1, \cdots, C_\nu$; let A' be a closed 2-manifold obtained by capping each boundary C_i with an open 2-cell Z_i [14]. Let Γ' denote the upper semi-continuous collection of continua consisting of the elements of Γ and points of Z_i for $i=1, \cdots, \nu$. Let T_1 be the monotone mapping carrying each $w \in A'$ onto that $g \in \Gamma'$ containing w, and M' the set Γ' topologized so that T_1 is continuous. We may regard M as $T_1(A)$. M' does not reduce to a point. Let us verify (2.1) (a) and (b) for every $g \in \Gamma'$. They obviously hold if $g \in \Gamma' - \Gamma$. Consider $g \in \Gamma$; g does not separate A', since g neither separates A nor contains any boundary C_j. Let G open in A be as in (2.1 a). If $G \cap A^*$ is void, set $G' = G$. Otherwise, using Lemma 3 there is a set G_1 open in A such that $g \subset G_1 \subset G$ and $G_1 \cap A^*$ is an arc of a single C_i. Let G' be an open subset of A' obtained by adjoining to G_1 a simply connected part of Z_i bounded by $G_1 \cap C_i$ and an arc in Z_i^-. Every closed curve in G' is homotopic to a closed curve in G_1, and so is null homotopic in A'. This shows that Γ' satisfies (a) and (b). Therefore, A' and M' are homeomorphic.

The restriction of T_1 to Z_j^- is one-one on Z_j for each j; and by Lemma 3, monotone on Z_j^-, not constant on Z_j^*. By [9, p. 65] $T_1(Z_j^-)$ is a 2-cell for each j, and by Lemma 3 these 2-cells are disjoint. This implies that M is a 2-manifold of the same topological type as A.

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[3] What has actually been shown is that if M is the image of a closed 2-manifold A under a monotone mapping T', $\Gamma(T', A)$ satisfies (2.1) (a) and (b), and M is not a point, then M and A are homeomorphic. This remark is used below

ANALYTICAL PART

4. Preliminaries. I write $a(P)$ for the elementary area of a polyhedron P and $L(S)$ for the Lebesgue area of a surface S, defined as the lower limit of elementary areas of nondegenerate polyhedra tending to S.

For the remainder of the paper we shall suppose A is a parallel slit domain (see §5), since this class will suffice for the problem of quasi-conformal representation. For the closed case $(\nu = 0)$, A will consist of the full plane including a point at ∞ with a finite number of semi-infinite identified slits parallel to the real axis; whereas for $\nu \geq 1$ A is an upper half plane with a finite number of slits of a similar sort.

It is convenient to represent any finite point of A by its (u, v) coordinates. On the slits the correspondence between points of A and points (u, v) is not biunique, but is determined by the identifications. A continuous mapping from A to E_N is simply any vector function $T(u, v) = [T^1(u, v), \cdots, T^N(u, v)]$ which is continuous in every closed region whose interior intersects no slit, takes the same value at identified points on the slits, and is continuous at ∞. We write

$$D(T, A) = \frac{1}{2} \int\int (E + G) du dv, \qquad I(T, A) = \int\int (EG - F^2)^{1/2} du dv$$

for the *Dirichlet integral* and *classical area integral*, respectively, whenever they exist. The integrals are taken over the full plane or upper half plane according as A is closed or not.

$T(u, v)$ is called a *Dirichlet* mapping if: (1) $T(u, v)$ defines a continuous mapping from A into E_N; (2) each component $T^i(u, v)$ is absolutely continuous in Tonelli's sense in every closed (u, v) interval disjoint from the slits of A, $i = 1, \cdots, N$; and (3) $D(T, A)$ is finite. $T(u, v)$ is *quasi-conformal*[4] if $T(u, v)$ is Dirichlet and $D(T, A) = I(T, A)$. The latter is equivalent to $E = G$, $F = 0$ almost everywhere.

5. Parallel slit domains. This section follows, in part, Shiffman [11].

(a) *Closed slit domains.* In a (u, v) plane let there be made β (≥ 2) slits parallel to the u-axis, semi-infinite to the right. The ordinates V_j of the slits are ordered so that $V_1 < \cdots < V_\beta$. The two sides of each slit are regarded as distinct. There are also given *critical abscissae* $U_1 < \cdots < U_\alpha$ ($\alpha \geq 1$). Let F denote the set of points (U_i, V_j), $i = 1, \cdots, \alpha$, $j = 1, \cdots, \beta$. The left endpoint of every slit shall belong to F, and F generally contains other points as well. Identifications are made as follows. Let s be any segment on a slit L, with endpoints in F or one endpoint in F and the other ∞. Then s is identified with exactly one other segment s', on another slit or on the opposite side of L, whose endpoints have the same abscissae as s. In fact, the identifi-

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(4) This term has been used by L. C. Young [12]. It is equivalent to the term *almost conformal* used by Cesari [1, p. 477], and to the term *generalized conformal* used by Radó [9, p. 480] and Morrey [7].

cation of s and s' is made pointwise, each pair of identified points having the same abscissa([5]). To insure that ∞ belongs to an open 2-cell, we must require that the identifications for abscissa u in the interval (U_α, ∞) suitably "interlock" [11].

This defines a closed slit domain A. The *order* of a finite point w on the slits is $a+b/2-1$, where a is the number of left endpoints of slits and b the number of other points corresponding to w. Those points with positive order are called *vertices* of A. A pair of identified segments joining vertices, or a vertex with ∞, define an *edge*. Clearly every vertex has critical abscissa.

(b) *Slit domains with ν ($\geqq 1$) boundaries.* The following modifications in (a) are needed. We consider the upper half plane with β ($\geqq 1$) semi-infinite slits with positive ordinates V_j and critical abscissae U_i, as before. F consists of the points (U_i, V_j), $i=1, \cdots, \alpha, j=0, 1, \cdots, \beta$, where we set $V_0=0$. A segment s may be identified with another segment on a slit as before, or with a segment on the real axis lying directly below s; or s may be unidentified. The unidentified segments of slits, together with the unidentified part of the real axis form the boundary A^*. Vertices not on A^* are defined as before, and are called *inner*. The order of a finite point $w \in A^*$ is $a+b/2-1/2$, and a point of A^* with positive order is a *boundary vertex*. Since at least part of the negative real axis is unidentified, ∞ is on A^*. The identifications for u in the interval (U_α, ∞) must be such that ∞ belongs to a half-open 2-cell. A *free edge* is a segment of a slit, or of the real axis, joining boundary vertices or a boundary vertex with ∞. If no part of the real axis is identified, it counts as a free edge. Every boundary vertex adjoins exactly two free edges.

A slit domain A is called *normalized* if: (1) The left endpoint of every slit corresponds to a vertex and every critical abscissa U_i is the abscissa of at least one vertex; (2) $U_1=0$, $V_\beta=1$, and if A is closed $V_1=0$([6]). Condition (1) excludes superfluous abscissae U_i as well as the possibility that every point of a slit L, or of a segment at the left end of L, is merely identified with the point on the opposite side of L. Any slit domain, not topologically a 2-sphere or 2-cell, can be normalized by deleting such superfluous abscissae and slits (or pieces of slits) followed by a linear transformation of the complex plane (or upper half plane) onto itself.

It is not excluded that some point p on L to the right of a vertex is identified with that point of L opposite p. For example, L may carry two free edges situated on opposite sides of a same finite segment t of L, every point of L to the right of t being identified with the point opposite.

(c) *Convergence of normalized slit domains*; (cf. [11, p. 861]). Let us say that two normalized slit domains are *similarly arranged* if the geometric configuration of the slits, vertices, and edges (as well as the manner of identifying

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([5]) This condition arises naturally in mapping a closed Riemann domain conformally onto a slit domain [3, Chapter 2].

([6]) This normalization replaces the 3-point condition used in proving Morrey's Theorem.

edges) is the same for both. Similarly arranged slit domains obviously are of the same topological type τ, but not conversely. If q denotes the characteristic, the sum of the orders of the vertices is $q+2$ or $q+1$ according as A is closed or not [11, pp. 858–859]. It follows that there are only finitely many different arrangements for domains of any given type τ.

By *distance* between two finite points w, w' of a slit domain A we mean the euclidean length of the shortest curve in A joining w and w'.

Let A_1, A_2, $\cdots$ be normalized slit domains, similarly arranged, such that corresponding points of F_n tend to finite limits as $n \to \infty$. The limit domain A is defined as follows. F is the set of limit points of F_n; i.e., the slits and critical abscissae of A are the limits of the slits and critical abscissae, respectively, of A_n. Several points of F_n may tend to the same point of F. The identifications in A are determined if we specify them at any point p on a slit whose abscissa u is not critical. If p is on the upper (lower) side of a slit L, for large n let p_n be that point with abscissa u on the upper side of the *highest* slit (lower side of the *lowest* slit) of A_n which tends to L. $p \in A^*$ if the distance in A_n between p_n and A_n^* tends to 0 as $n \to \infty$. p is identified with another point p' with abscissa u if the distance in A_n between p_n and p_n' tends to 0.

With these definitions A is a slit domain, not necessarily normalized, possibly with fewer boundaries or lower characteristic than A_n. We shall next determine conditions under which A and A_n are homeomorphic.

(d) *Some curves in* $\Omega(A)$. If A is a normalized slit domain the following types of curves belong to $\Omega(A)$:

(5.1) A closed curve or cross-cut lying entirely on a finite portion of a single vertical line. No such curve separates A.

(5.2) A component C_i of A^* (we assume A is not a 2-cell).

(5.3) A simple closed curve γ formed by joining identified points p and p' by an arc not meeting any slit between endpoints nor passing through ∞. If p and p' are on different slits, γ does not separate A. If γ separates, then each component of $A - \gamma$ must contain a component of A^* and we apply Lemma 1.

(5.4) A simple closed curve formed by joining identified points p and p' on an identified edge whose right endpoint is a vertex (i.e., not ∞) with ∞ by a pair of arcs meeting no slit between endpoints. See [11, p. 859].

(5.5) A simple closed curve γ passing through ∞ encircling at least one slit (but not all slits if A is closed) and meeting no slit at any finite point. Either γ does not separate; or if it does, each component of $A - \gamma$ contains in its closure a closed curve of one of the previous types.

(e) *Dissection of slit domains by gratings.* Let π be a grating composed of the perimeter R^* of an interval R in the u, v plane together with a finite number of segments parallel to the coordinate axes with endpoints on R^*. We say that π is based on R. $[\pi]$ is the set in the plane occupied by π. $\Delta(\pi)$, $\overline{\Delta}(\pi)$ denote, respectively, the minimum and maximum distance between

adjacent parallel lines of π. Let A be a slit domain whose slits are disjoint from the horizontal lines of π and whose critical abscissae are distinct from the abscissae of vertical lines of π. K denotes the set of points in A with (u, v) coordinates in $[\pi]$. We say that π is *fine* relative to A if $F \subset R$ and $2\overline{\Delta}(\pi)$ is less than min $(U_{i+1} - U_i)$ and min $(V_{j+1} - V_j)$. If π is fine every component of $A - K$ is an open or half open 2-cell. If A carries a subscript n the sets F and K will carry that same subscript.

In Lemma 4 (as well as certain later sections) the following notation applies. Let A_n tend to A, and π a grating fine relative to A. We dissect both A and A_n with π, for large n. The grating π need not be fine relative to A_n; see examples in [3] and [11]. $d > 0$ is a number not exceeding the distance between $[\pi]$ and F. Let $n(d)$ be such that the set F_n is in the $d/2$ neighborhood of F for every $n \geq n(d)$. In particular, for $n \geq n(d)$ every slit of A_n is distant at most $d/2$ from some slit of A. $Z_1, \cdots, Z_k$ are the components of $A - K$; $\gamma_j = Z_j^* \cap K$ the relative boundary of Z_j; $w_j = (u_j, v_j)$ a point of γ_j distant at least $d/2$ from any slit of A; and w_{jn} the point of A_n with coordinates (u_j, v_j), $j = 1, \cdots, k$.

LEMMA 4. *Choose $n \geq n(d)$. Suppose that, if Z is any component of $A_n - K_n$, then its relative boundary $Z^* \cap K_n$ contains no simple closed curve or cross-cut $\gamma \in \Omega(A_n)$. Then there is a homeomorphism H_n of A onto A_n such that $H_n(w_j) = w_{jn}$ and $H_n(Z_j)$ is a component of $A_n - K_n$ for $j = 1, \cdots, k$. In particular, A and A_n are of the same topological type.*

Proof. Let γ_{jn} denote the following simple closed curve or cross-cut lying in K_n, homeomorphic with γ_j. Outside the $d/2$ neighborhood of the slits of A, γ_{jn} and γ_j are superimposed. Each piece λ of the intersection of γ_j with the $d/2$ neighborhood is either a vertical line ending at a point p on the boundary A^* or a pair of vertical lines, one ending at a point p on an identified edge, the other at the point p' identified with p. In the first case the corresponding λ_n is a vertical curve with the same starting point as λ which proceeds to p_n (notation of (c) above) and then joins p_n following the identifications in A_n to A_n^*. The second case is similar.

Clearly γ_{jn} is all or part of the relative boundary of some component of $A_n - K_n$. Then by hypothesis and Lemma 1, γ_{jn} separates A_n and moreover one of the two components of $A_n - \gamma_{jn}$ is an open or half-open 2-cell. Let Z_{jn} be that component corresponding naturally to Z_j, i.e., superimposed on Z_j except at most in the $d/2$ neighborhood of the slits of A. $Z_{1n}, \cdots, Z_{kn}$ are disjoint since $\gamma_{in} \neq \gamma_{jn}$ for $i \neq j$. The complement of Z_{jn} contains a closed curve of type (5.3) or (5.5) according as Z_{jn} contains ∞ or not, since π is fine relative to A. Hence, Z_{jn}^- is a 2-cell for each j. Moreover, there are no other components of $A_n - K_n$ since any such component must contain in its relative boundary a curve of type (5.1); cf. [11, p. 862].

We define H_n first as a homeomorphism from K onto K_n, such that

$H_n(\gamma_j) = \gamma_{jn}$ and $H_n(w_j) = w_{jn}$, $j = 1, \cdots, k$. By a known theorem $[1, \text{p. } 55]$ H_n has an extension as a homeomorphism from Z_j^- onto Z_{jn}^- for $j = 1, \cdots, k$.

6. **Continuation.** Let Q_a denote the square with center the origin, side length a, and sides parallel to the u, v axes. The following is a variant of an important lemma due to L. C. Young $[12, \text{p. } 318]$; the proof is almost the same as Young's.

LEMMA 5 ($\epsilon\delta$ *grating*). *Given $M > 0$ and $\epsilon > 0$ there exists $\eta = \eta(M, \epsilon) > 0$ with the following property. If $T(u, v)$ is any Dirichlet mapping from a slit domain A into E_N with $D(T, A) \leq M$ and $a > 0$ is arbitrary, there exists δ with $\eta < \delta < \epsilon$ and a grating π based on Q_a such that $a\delta < \Delta(\pi) \leq \bar{\Delta}(\pi) < 2a\delta$ and, if γ is any vertical or horizontal side not on Q_a^* of a rectangle formed by the grating π, then T represents on γ a finite sum of curves of total length $< \epsilon$.*

Let T_n be defined in the slit domain A_n for $n = 1, 2, \cdots$. We call the sequence T_n *equicontinuous* if: (1) for every $a > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that $|T_n(w) - T_n(w')| < \epsilon$ if w and w' are distant in A_n (cf. §5(c)) less than δ and both w and w' lie inside the square Q_a, $n = 1, 2, \cdots$; and (2) given $\epsilon > 0$ there exists $a > 0$ such that $|T_n(w) - T_n(\infty)| < \epsilon$ for all w outside Q_a and $n = 1, 2, \cdots$ (equicontinuity at ∞).

LEMMA 6. *Let S_n tend to a nondegenerate surface S, and let $T_n(u, v)$ be a Dirichlet representation of S_n on a normalized slit domain A_n with $D(T_n, A_n)$ bounded by a constant M independent of n. Then there is a subsequence of n for which A_n tends to a limit A homeomorphic with A_n and T_n is equicontinuous.*

Proof. For definiteness suppose A_n has boundary. If A_n is closed the proof is similar and slightly simpler. There are only finitely many possible arrangements of A_n, all A_n being of the same topological type. We may select a subsequence (denoted again by T_n, A_n) for which all A_n are similarly arranged and corresponding points of F_n tend to finite limits or to ∞.

Let us first show that no points of F_n tend to ∞. Choose positive

$$\epsilon < \phi(S)/24$$

(see Lemma 2) and $\eta = \eta(M, \epsilon)$. Let a be chosen so that all finite limit points of F_n are interior to the square $Q_{\eta a}$. Choose n_0 so that for $n \geq n_0$, $\phi(S_n) \geq \phi(S)/2$ and each point of F_n either lies inside $Q_{\eta a}$ or outside Q_a. Consider fixed $n \geq n_0$ and $\epsilon\delta$ grating π based on Q_a with $T(u, v) = T_n(u, v)$ and $A = A_n$ in Lemma 5. Let R be a rectangle composed of a rectangle R' formed by π whose closure contains the origin, together with the 8 rectangles contiguous to R'. $Q_{\eta a} \subset R$ and $R \subset Q_a$ if $\epsilon < 1/8$. Let B_n be the part of A_n inside R and γ_n the relative boundary of B_n. The T_n image of each component of γ_n has diameter $< 12\epsilon < \phi(S_n)$; therefore no such component belongs to $\Omega(A_n)$. It follows that γ_n is a cross-cut with both endpoints on that component of A_n^* containing ∞ (γ_n certainly meets this component on the negative real

axis since all slits are in the right half plane). By Lemma 1 either B_n^- or $A_n - B_n$ is a 2-cell.

Since A_n is normalized, B_n contains at least one vertex w. B_n meets some identified edge; otherwise w is the left endpoint of two free edges lying on opposite sides of the same slit and both meeting γ_n. Since γ_n is a cross-cut also meeting A_n^* on the negative real axis this is impossible. Then B_n contains a closed curve of type (5.3), which proves that $A_n - B_n$ must be a 2-cell. Therefore, $A_n - B_n$ contains no curve of type (5.4), which implies that no point of F_n is in $A_n - B_n$. Thus, all points of F_n tend to finite limits and the limit domain A is defined. .

Let us next show that A and A_n are homeomorphic. Let β denote the number of slits of A. Choose a so that $F \subset Q_{a/2}$, and $\epsilon < \phi(S)/24(\beta+1)$ small enough that any grating π based on Q_a with $\overline{\Delta}(\pi) < 6\epsilon a$ is fine relative to A. Let $\eta_1 = \eta(M, \epsilon)$ and $d = \eta_1 a$. Fix $n \geq n(d)$ large enough that $\phi(S_n) \geq \phi(S)/2$ and let π_0 be an $\epsilon\delta$ grating based on Q_a with $T = T_n$ and $A = A_n$ in Lemma 5. Let π be a subgrating obtained by deleting all lines in π_0 distant at most d from F together with part of π_0 in a certain neighborhood of Q_a^*. Then $\overline{\Delta}(\pi) < 6\delta a < 6\epsilon a$.

The unbounded component of $A_n - K_n$ is a 2-cell with a cross-cut as relative boundary. Every bounded component not meeting any slit is a rectangle. If Z is any bounded component meeting some slit, and γ any component of the relative boundary of Z, then

$$\text{diam } T_n(\gamma) < 12(\beta + 1)\epsilon < \phi(S_n).$$

Hence, $\gamma \notin \Omega(A_n)$. By Lemma 4, A and A_n are homeomorphic.

Finally, we have to show that the sequence T_n is equicontinuous. Suppose not. Then there exist $\epsilon^* > 0$, a subsequence of n, and for n in this subsequence $w_n, w_n' \in A_n$ such that $|T_n(w_n) - T_n(w_n')| > \epsilon^*$ and either the distance in A_n of w_n and w_n' tends to 0 while w_n and w_n' remain in a fixed square $Q_{a/2}$ containing F or else w_n, w_n' tend to ∞. We may assume $\epsilon^* < \phi(S)/2$. Consider the first case. By Lemma 2 and (2.3) there exist $c > 0$ and $n_1 \geq n_0$ such that $\psi(S_n, \epsilon^*) \geq c$ for all $n \geq n_1$. Choose $\epsilon < c/20(\beta+1)$ small enough that any grating π based on Q_a with $\overline{\Delta}(\pi) < 10\epsilon a$ is fine relative to A and $n \geq n_1$ so that w_n and w_n' are distant $< \eta_1 a/2$. We construct again an $\epsilon\delta$ grating. This time not only lines in π_0 distant at most $\eta_1 a$ from F but also those distant in A_n at most $\eta_1 a$ from w_n are deleted. Then w_n, w_n' belong to the same bounded component Z of $A_n - K_n$; and if we suppose $c < \phi(S_n)$ (as we may) then Z^- is a 2-cell with relative boundary γ a cross-cut or simple closed curve with diam $T_n(\gamma) < c$. Moreover, there exists $\lambda \in \Omega(A_n)$ contained in $A_n - Z^-$. Then

$$\text{diam } T_n(A_n - Z^-) \geq \text{diam } T_n(\lambda) \geq \phi(S_n) > \epsilon^*,$$

and

$$\text{diam } T_n(Z) \geq |T_n(w_n) - T_n(w_n')| > \epsilon^*.$$

Hence, $\psi(S_n, \epsilon^*) \leq \text{diam } T_n(\gamma) < c$, a contradiction.

In case w_n and w_n' tend to ∞ we define γ_n as in the first part of the proof so that diam $T_n(\gamma_n) < c$. For large n, B_n contains some $\lambda \in \Omega(A_n)$ while $A_n - B_n^-$ contains w_n, w_n'. As before this is impossible.

7. The main theorem.

LEMMA 7. *Every polyhedron (not of the type of the 2-cell or 2-sphere) has a quasi-conformal representation on a normalized parallel slit domain.*

This is a special case of mapping theorems for general Riemann domains proved in [3, Chapter II].

LEMMA 8. *Let T be Dirichlet in a slit domain A and S the Fréchet surface represented by T. Then $L(S) = I(T, A)$; i.e., the Lebesgue area equals the classical area integral.*

Proof. Let π be a grating fine relative to A such that T represents a rectifiable curve on every arc of the associated set K. Now K decomposes A into nonoverlapping 2-cells $Z_1^-, \cdots, Z_m^-$ disjoint from A^* and annular regions $Y_1, \cdots, Y_\nu$ formed by 2-cells $Z_{m+1}^-, \cdots, Z_k^-$ which meet A^*. For $j = 1, \cdots, \nu$, Y_j is bounded by C_j and a polygonal curve in K. For $i = 1, \cdots, m$ Z_i is either a rectangle (possibly formed by joining two euclidean rectangles along an identified edge) or else contains ∞ or an inner vertex. In any case, Z_i^- is the image of a planar 2-cell W_i under a one-one elementary transformation ψ_i, conformal except for one point at most. By known theorems, $T\psi_i$ is Dirichlet in W_i and $I(T\psi_i, W_i) = I(T, Z_i)$, $i = 1, \cdots, m$. Then if S_i is the surface represented by (T, Z_i^-), $L(S_i) = I(T, Z_i)$ by the known analogue of Lemma 11 for planar 2-cells [1, p. 44].

Given $\epsilon > 0$ choose π so that, writing $A_\epsilon = \bigcup_{i=1}^m Z_i^-$, we have $I(T, A) - I(T, A_\epsilon) < \epsilon$ and diam $T(Z_i) < \epsilon/2$ for $i = m+1, \cdots k$. If S_ϵ is that surface represented by (T, A_ϵ), then $L(S_\epsilon) = \sum_{i=1}^m L(S_i)$ by a result of Federer [6]. Moreover, $L(S_\epsilon) \leq L(S)$ [5, (6.25)]. There is a homeomorphism H_ϵ of A_ϵ onto A such that $H_\epsilon(w) = w$ if w belongs to a slightly smaller subdomain interior to A_ϵ and $|T[H_\epsilon(w)] - T(w)| < \epsilon$ for all remaining $w \in A_\epsilon$. This shows that $\|S, S_\epsilon\| < \epsilon$. Hence, $\lim_{\epsilon \to 0} L(S_\epsilon) = L(S)$ since $L(S_\epsilon) \leq L(S)$ and Lebesgue area is lower semi-continuous. However,

$$L(S_\epsilon) = \sum_{i=1}^m L(S_i) = \sum_{i=1}^m I(T, Z_i) = I(T, A_\epsilon)$$

and the right side tends to $I(T, A)$.

MAIN THEOREM. *Every nondegenerate Fréchet surface S of any finite topological type τ (not the type of the 2-cell or 2-sphere([7])) with finite Lebesgue area has a quasi-conformal representation on a normalized slit domain.*

([7]) The theorem is true for 2-cells and 2-spheres provided we count the half plane and full plane with no slits as slit domains. This follows from Morrey's theorem [7].

Proof. Let P_n be a sequence of polyhedra tending to S such that $a(P_n)$ tends to $L(S)$. By Lemma 7 each P_n has a quasi-conformal representation $T_n(u, v)$ on a normalized slit domain A_n; $D(T_n, A_n) = a(P_n)$ is bounded.

We apply Lemma 6. For any compact region G disjoint from the slits of A, there is a further subsequence such that $T_n(u, v)$ converges to a limit $T(u, v)$ uniformly in G, by Ascoli's theorem. Diagonalizing, there is a still further subsequence, again denoted by (T_n, A_n), independent of G such that $T_n(u, v)$ tends to $T(u, v)$ uniformly in every such G. By equicontinuity, T is continuous across slits of A and at ∞. By known theorems $[1, 32.7]$ $T(u, v)$ is absolutely continuous Tonelli in G and $D(T, G) \leq \lim\inf D(T_n, G)$ for every such G. This implies that T is Dirichlet in A and

$$D(T, A) \leq \lim\inf D(T_n, A_n) = \lim a(P_n) = L(S).$$

Let us show that (T, A) represents S. For this we use Lemma 4, and follow the notation there. Given $\epsilon > 0$, we choose π so fine that diam $T(Z) < \epsilon/3$ and diam $T_n[H_n(Z)] < \epsilon/3$ for any $n \geq n(d)$ and component Z of $A - K$. This is possible by equicontinuity and the remark that the diameter of every bounded component of $A_n - K_n$ (in the A_n metric) is uniformly small with $\overline{\Delta}(\pi)$. For some $n_1 \geq n(d)$, $|T_n(w_{jn}) - T(w_j)| < \epsilon/3$ for all $n \geq n_1$ and $j = 1, \cdots, k$. Then $|T_n[H_n(w)] - T(w)| < \epsilon$ for all $w \in A$, $n \geq n_1$. Since ϵ is arbitrary this shows that (T, A) represents $\lim P_n$ which is S.

Finally, $L(S) = I(T, A)$ by Lemma 8. Since $I(T, A) \leq D(T, A)$ and $D(T, A) \leq L(S)$, we have $D(T, A) = I(T, A) = L(S)$. Thus, T is quasi-conformal.

8. **Final remarks.** In the above theorem slit domains can be replaced by domains of various other forms. For example, let S be nondegenerate of genus 0 (i.e., of the type of a finitely connected planar Jordan region). Let (T, A) represent S quasi-conformally, A a slit domain. There is a one-one conformal mapping f of a planar region J bounded by v circles onto A $[3, p. 169]$. It can be shown that (Tf, J) is also a quasi-conformal representation of S. For other possible classes of domains in case of genus not 0, see $[3, \text{Chapters } 4, 5; 4]$.

AN EXAMPLE. Let J be an annulus bounded by two concentric circles, and g a segment joining opposite boundaries of J. One can construct a mapping $T(w)$ from J into E_3 with the following properties: (1) T is constant on g; (2) for $w_1, w_2 \notin g$, $w_1 \neq w_2$, we have $T(w_1) \neq T(w_2) \neq T(g)$; (3) given $w \notin g$ and $a > 0$ there is a neighborhood G of g such that T represents a curve of length $\geq a$ on any curve joining w and G; (4) (T, J) represents a surface S of finite Lebesgue area. Intuitively, S is an annulus in E_3 pinched together at a single point which cannot be approached on S by any rectifiable curve. It is not difficult to show that S has no Dirichlet representation, hence no quasi-conformal representation. On the other hand, the fine cyclic decomposition $[2; 8]$ of S has only one element.

If J is cut along g and opposite sides of the cut are regarded as distinct, the resulting domain J' is a 2-cell and (T, J') represents a nondegenerate surface S' with the same Lebesgue area as S. By Morrey's theorem S' has a quasi-conformal representation. This sort of reasoning will be used in a subsequent paper to obtain representations of fine cyclic elements.

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